Symmetry Class of Tensors Associated with Certain Groups

A Thesis Submitted to Faculty of Science
University of Tehran

By
M. R. Pournaki

In Partial
Fulfillment of the Requirements for the
Degree of Doctor of Philosophy (Ph.D.) in
Pure Mathematics

Supervisor
Professor M. R. Darafsheh

Preface

In the last two decades, the theory of symmetry class of tensors has been one of the attractive subjects in the multilinear algebra. So many mathematicians have been working on the variety of problems concerning symmetry class of tensors. Below we have a glimpse on the main ideas of the thesis.

Let V be an m-dimensional vector space over the complex field $\mathbb{C}$, and let G be a subgroup of the symmetric group on n letters, $\$_n$, and let χ be an irreducible complex character of G. Suppose $\phi : \overset{n}{\times} V \to U$ be an n-linear function, where U is a finite dimensional vector space over $\mathbb{C}$. We say ϕ is *symmetric* with respect to G and χ if for all $v_1, \ldots, v_n \in V$,

$$\frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) \phi(v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)}) = \phi(v_1, \ldots, v_n).$$

A finite dimensional vector space S over $\mathbb{C}$ is called a *symmetry class of tensors* associated with G and χ if there is an n-linear function $\phi : \overset{n}{\times} V \to S$ which is symmetric with respect to G and χ such that

(i) $\langle Im\ \phi \rangle = S$,

(ii) for each finite dimensional vector space U over $\mathbb{C}$ and for each n-linear function $\psi : \overset{n}{\times} V \to U$, symmetric with respect to G and χ, there exists a unique linear transformation $f : S \to U$ such that the following diagram commutes.

$$\begin{array}{ll} \overset{n}{\times} V & \overset{\phi}{\longrightarrow} S \\ \psi \downarrow & \swarrow f \\ U & \end{array}$$

We can prove that the symmetry class of tensors associated with G and χ exists and it is unique up to isomorphisms of vector spaces. Also we can prove that the symmetry class of tensors associated with G and χ is (isomorphic with) the image of the linear operator $T(G, \chi) : \overset{n}{\otimes} V \to \overset{n}{\otimes} V$, where

$$T(G, \chi)(v_1 \otimes \cdots \otimes v_n) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)},$$

where $\overset{n}{\otimes}V$ is the n-th tensor power of V and $v_1 \otimes \cdots \otimes v_n$ is a typical element of it. We denote the symmetry class of tensors associated with G and χ by $V_\chi^n(G)$. In addition, if we assume that V is an m-unitary space, then $V_\chi^n(G)$ will become a unitary space.

Finding an explicit formula for the dimension of $V_\chi^n(G)$, investigation of the existence of an orthogonal basis of decomposable symmetrized tensors for $V_\chi^n(G)$ and non-vanishing of $V_\chi^n(G)$, for a general or certain group G, are open problems in this branch of mathematics. In this thesis, we consider certain groups G and an irreducible character χ of G and answer the above questions. The certain groups which we considered are $G = \langle \pi_1 \ldots \pi_p \rangle$, where the π_i's, $1 \leq i \leq p$, are disjoint cycles in $\$_n$, $G = T_{4n}$, $G = PSL_2(q)$ and a group G of order n as a subgroup of $\$_n$ with Cayley representation. The results of this thesis appeared in some international journals (see [2], [3], [4] and [5]).

M. R. Pournaki
March 2000

Extended Abstract

1 Introduction

Let V be an m-dimensional vector space over the complex field $\mathbb{C}$. Let $\overset{n}{\otimes}V$ be the n-th tensor power of V and write $v_1\otimes\cdots\otimes v_n$ for the decomposable tensor product of the indicated vectors. To each permutation σ in $\$_n$ there corresponds a unique linear operator $P(\sigma):\ \overset{n}{\otimes}V \to \overset{n}{\otimes}V$ determined by $P(\sigma)(v_1\otimes\cdots\otimes v_n) = v_{\sigma^{-1}(1)}\otimes\cdots\otimes v_{\sigma^{-1}(n)}$. Let G be a subgroup of $\$_n$ and $I(G)$ be the set of all the irreducible complex characters of G. It follows from the orthogonality relations for characters that

$$\Big\{T(G,\chi):\ \overset{n}{\otimes}V \to \overset{n}{\otimes}V|\ \ T(G,\chi) = \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)P(\sigma);\ \chi\in I(G)\Big\}$$

is a set of annihilating idempotents which sum to the identity. The image of $\overset{n}{\otimes}V$ under $T(G,\chi)$ is called the *symmetry class of tensors* associated with G and χ and it is denoted by $V_\chi^n(G)$. The image of $v_1\otimes\cdots\otimes v_n$ under $T(G,\chi)$ is denoted by $v_1 * \cdots * v_n$ and it is called a *decomposable symmetrized tensor.* It is well-known that

$$\dim V_\chi^n(G) = \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)m^{c(\sigma)} \tag{1}$$

where $c(\sigma)$ is the number of cycles, including cycles of length one, in the disjoint cycle decomposition of σ (see [9]). Also we have that

$$\bigotimes^n V = \bigoplus_{\chi\in I(G)} V_\chi^n(G) \tag{2}$$

is a direct sum.

Let Γ_m^n be the set of all sequences $\alpha = (\alpha_1,\ldots,\alpha_n)$ with $1 \le \alpha_i \le m$, so α is a mapping from a set of n elements into a set of m elements. Then the group G acts on Γ_m^n by $\sigma.\alpha = (\alpha_{\sigma^{-1}(1)},\ldots,\alpha_{\sigma^{-1}(n)})$ where $\sigma\in G$ is a permutation on n letters and $\alpha\in\Gamma_m^n$ is a mapping from a set of n elements into a set of m elements. Therefore the action may be written as $\sigma.\alpha = \alpha\sigma^{-1}$ which is a composition of two functions. Let $O(\alpha) = \{\sigma.\alpha|\ \sigma\in G\}$ be the *orbit* with representative α, also let G_α be the *stabilizer* of α, i.e., $G_\alpha = \{\sigma\in G|\ \sigma.\alpha = \alpha\}$. Let Δ be a system of distinct representatives of the orbits of G acting on Γ_m^n and define

$$\overline{\Delta} = \Big\{\alpha \in \Delta | \sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0\Big\},$$

and let Ω be the union of those equivalence classes represented by elements of $\overline{\Delta}$.

Let $\{e_1, \ldots, e_m\}$ be a basis of V. Denote by e^*_α the tensor $e_{\alpha_1} * \ldots * e_{\alpha_n}$ where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Gamma^n_m$. For $\alpha \in \overline{\Delta}$, $V^*_\alpha = \langle e^*_{\sigma.\alpha} | \ \sigma \in G\rangle$ is called the *orbital subspace* of $V^n_\chi(G)$. It follows that

$$V^n_\chi(G) = \bigoplus_{\alpha \in \overline{\Delta}} V^*_\alpha, \tag{3}$$

is a direct sum. In [6] Freese proved that

$$\dim V^*_\alpha = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma), \tag{4}$$

in particular, if χ is of degree one, then $\dim V^*_\alpha = 1$ for all $\alpha \in \overline{\Delta}$.

A particular case appear, when we assume that V is an m-unitary space. In this case the inner product on V induces an inner product on $\overset{n}{\otimes} V$ whose restriction to $V^n_\chi(G)$ satisfies

$$\langle u_1 * \cdots * u_n \mid v_1 * \cdots * v_n\rangle = \frac{\chi(1)}{|G|} d^G_\chi(A)$$

where $A = [a_{ij}]_{n\times n} = [\langle u_i | v_j\rangle]_{n\times n}$ and $d^G_\chi(A) = \sum_{\sigma \in G} \chi(\sigma) a_{1\sigma(1)} \ldots a_{n\sigma(n)}$ is the *generalized matrix function.*

With respect to this inner product the sums appearing in (2) and (3) are orthogonal direct sums. Also if we suppose $\{e_1, \ldots, e_m\}$ is an orthonormal basis of V, then we obtain

$$\langle e^*_\alpha \mid e^*_\beta\rangle = \begin{cases} \frac{\chi(1)}{|G|} \sum\limits_{\sigma \in G_\beta} \chi(\sigma\tau^{-1}) & \text{if} \quad \alpha = \tau.\beta \text{ for some } \tau \in G, \\ \\ 0 & \text{if} \quad O(\alpha) \neq O(\beta). \end{cases}$$

In particular, taking the norm of e^*_α, with respect to the induced inner product, one easily obtains the condition $e^*_\alpha \neq 0$ if and only if $\alpha \in \Omega$.

If $\alpha = g.\gamma$ and $\beta = g'.\gamma$, then $gg'^{-1}.\beta = \alpha$, so if we set $\tau = gg'^{-1}$ and use the above formula for $\langle e^*_\alpha \mid e^*_\beta \rangle$, then we obtain

$$\langle e^*_{g.\gamma} \mid e^*_{g'.\gamma} \rangle = \frac{\chi(1)}{|G|} \sum_{\sigma \in G_\gamma} \chi(g'\sigma g^{-1}). \tag{5}$$

An orthogonal basis of the form $\{e^*_\alpha | \ \alpha \in S\}$, where S is a subset of Γ^n_m, is called an O-basis for $V^n_\chi(G)$. By (3), $V^n_\chi(G)$ has an O-basis if and only if for all $\alpha \in \overline{\Delta}$, the orbital subspace V^*_α has an O-basis. In particular, if χ is of degree one, since $\dim V^*_\alpha = 1$ for all $\alpha \in \overline{\Delta}$, then V^*_α has an O-basis for all $\alpha \in \overline{\Delta}$ which implies that $V^n_\chi(G)$ has such a basis. Several papers are devoted in investigation of the non-vanishing of $V^n_\chi(G)$ and finding a formula for $\dim V^n_\chi(G)$ in a more closed form than (1), also discuss about the existence of an O-basis for these vector spaces, see for example [1], [7] and [14]. In [10] and [12] a formula for $\dim V^n_\chi(G)$ is given when G is equal to the whole group of $\$_n$, also in [13] a formula for calculating $\dim V^n_\chi(G)$ in the case that $G = \langle \pi_1 \rangle \cdots \langle \pi_p \rangle$ is given, where the π_i's, $1 \leq i \leq p$, are disjoint cycles in $\$_n$. Also, in [7] a necessary and sufficient condition for the existence of O-basis for $V^n_\chi(G)$ is given, when G is a cyclic or a dihedral group. The books [8] and [11] are good sources of information about characters of finite groups and the symmetry class of tensors respectively.

2 On the Dimensions of Symmetry Classes of Tensors Associated with the Group $G = \langle \pi_1 \ldots \pi_p \rangle$

Abstract

The dimensions of the symmetry classes of tensors, associated with a certain cyclic subgroup of $\$_n$ which is generated by a product of disjoint cycles is explicitly given in terms of the generalized Ramanujan sum. These dimensions can also be expressed as the Euler φ-function and the Möbius function. In the following we show some results appeared in [2].

Definition 2.1 Let $n_1, \ldots, n_p$ be positive integers and let h be a nonnegative integer. Suppose $d_1 | n_1, \ldots, d_p | n_p$. The *generalized Ramanujan sum* denoted by $S(h; n_1, \ldots, n_p; d_1, \ldots, d_p)$ is defined by

$$S(h;n_1,\ldots,n_p;d_1,\ldots,d_p)=\sum_{\substack{t=0\\(t,n_1)=d_1\\ \vdots\\(t,n_p)=d_p}}^{[n_1,\ldots,n_p]-1}\exp\left(\frac{2\pi iht}{[n_1,\ldots,n_p]}\right).$$

If the set $\{0\le t\le[n_1,\ldots,n_p]-1|\ (t,n_i)=d_i;\ 1\le i\le p\}$ is empty, then we define $S(h;n_1,\ldots,n_p;d_1,\ldots,d_p)=0$.

Lemma 2.2 *Let $n_1,\ldots,n_p$ be positive integers and let h be a nonnegative integer. Suppose $d_1|n_1,\ldots,d_p|n_p$ and set $n'_i=n_i/d_i$, $N_i=n_1\ldots n_p/n_i$, $N'_i=n'_1\ldots n'_p/n'_i$, $D_i=d_1\ldots d_p/d_i$ $(1\le i\le p)$ and*

$$l=\frac{(N_1,\ldots,N_p)}{(N'_1,\ldots,N'_p)(D_1,\ldots,D_p)}.$$

Then we have

$$S(h;n_1,\ldots,n_p;d_1,\ldots,d_p)=\begin{cases}\frac{1}{l}C_{[n'_1,\ldots,n'_p]}(hl) & ,\quad \textit{if}\ \left(\frac{[d_1,\ldots,d_p]}{d_i},n'_i\right)=1, \\ & \qquad\qquad 1\le i\le p\\ 0 & ,\quad \textit{otherwise.}\end{cases}$$

In this section, as we mentioned earlier, the group $G=\langle\pi_1\ldots\pi_p\rangle$ is considered, where the π_i's, $1\le i\le p$, are disjoint cycles in $\$_n$. Our main result, which appears in the following theorem, is to calculate $\dim V^n_\chi(G)$, where $\chi\in I(G)$, in terms of known number theoretical functions. Our formula involves the generalized Ramanujan sum.

Theorem 2.3 *Let $G=\langle\pi_1\ldots\pi_p\rangle$, where the π_i's, $1\le i\le p$, are disjoint cycles in $\$_n$ of orders $n_1,\ldots,n_p$, respectively, and let χ_h, $0\le h\le[n_1,\ldots,n_p]-1$, be an irreducible complex character of G. If V be an m-dimensional vector space over $\mathbb{C}$, then*

$$\dim V^n_{\chi_h}(G)=\frac{m^{n-(n_1+\cdots+n_p)}}{[n_1,\ldots,n_p]}\sum_{\substack{d_1|n_1\\ \vdots\\ d_p|n_p}}S(h;n_1,\ldots,n_p;d_1,\ldots,d_p)m^{d_1+\cdots+d_p}$$

where $S(h;n_1,\ldots,n_p;d_1,\ldots,d_p)$ denotes the generalized Ramanujan sum.

3 On the Orthogonal Basis of the Symmetry Classes of Tensors Associated with the Dicyclic Group

Abstract

A necessary and sufficient condition for the existence of O-basis for the symmetry classes of tensors associated with the dicyclic group is given. In particular we apply these conditions to the generalized quaternion group, for which the dimensions of the symmetry classes of tensors are computed. In the following we show some results appeared in [3].

Let $G = T_{4n} = \langle r, s | \; r^{2n} = 1, r^n = s^2, s^{-1}rs = r^{-1} \rangle, \; n \geq 1$, be the dicyclic group as a subgroup of $\$_{4n}$ with Cayley representation, and let V be an m-unitary space, with orthonormal basis $\{e_1, \ldots, e_m\}$. For $n = 1$, the dicyclic group T_4 is cyclic, $T_4 \simeq \mathbb{Z}_4$, therefore all of its irreducible characters are of degree 1 and so $V_\chi^4(T_4)$ has an O-basis for all $\chi \in I(T_4)$. Therefore we assume that $n \geq 2$. If $m = 1$, then $\dim \overset{4n}{\otimes} V = 1$, so $\dim V_\chi^{4n}(G) = 0$ or 1, therefore we do not have any problem about the existence of O-basis for $V_\chi^{4n}(G)$ for all $\chi \in I(G)$, therefore we assume that $m \geq 2$.

For irreducible characters of T_{4n} of degree 1, $\psi_i, \; \phi_i, \; 1 \leq i \leq 4$, $V_{\psi_i}^{4n}(T_{4n})$ and $V_{\phi_i}^{4n}(T_{4n})$ have O-basis and so we do not deal with the ψ_i's and ϕ_i's.

Therefore we investigate the problem for irreducible characters of degree 2 of T_{4n}, i.e., $\chi_h, \; 1 \leq h \leq n-1$, which are given by

$$\chi_h(r^k) = 2\cos\frac{kh\pi}{n}, \qquad \chi_h(r^k s) = 0, \qquad 0 \leq k < 2n.$$

The following results are the main results in this section.

Theorem 3.1 *Let $G = T_{4n}, \; n \geq 2$, and $\chi = \chi_h, \; 1 \leq h \leq n-1$, $\dim V = m \geq 2$. Then $V_\chi^{4n}(G)$ has an O-basis if and only if $\nu_2(\frac{h}{n}) < 0$, where ν_2 is 2-adic valuation.*

Corollary 3.2 *Let $G = T_{4n}, \; n \geq 2$ is odd, and $\chi = \chi_h, \; 1 \leq h \leq n-1$, $\dim V = m \geq 2$. Then $V_\chi^{4n}(G)$ does not have an O-basis.*

Theorem 3.3 *Let $G = Q_{2^{n+1}} = T_{4(2^{n-1})}, \; n \geq 2$, the generalized quaternion group, and $\chi = \chi_h, \; 1 \leq h \leq 2^{n-1} - 1$, $\dim V = m \geq 2$. Then $V_\chi^{2^{n+1}}(G)$ has an O-basis.*

4 On the Non-Vanishing of the Symmetry Classes of tensors Associated with $G = PSL_2(q)$

Abstract

The dimensions of the symmetry classes of tensors associated with the projective special linear group of degree 2 over a field with q elements, $PSL_2(q)$, are found. Of course we will assume $PSL_2(q)$ as a subgroup of the symmetric group $\$_{q+1}$ because this group has a faithful action on the points of the underlying projective space. We also discuss the non-triviality of the symmetry classes of tensors associated with each irreducible character of $PSL_2(q)$. In the following we show some results appeared in [4].

It is well-known that $G = PSL_2(q)$ acts faithfully and 2-transitively on the $q+1$ points of the projective line Ω and so we can assume that $G = PSL_2(q)$ is a subgroup of $\$_{q+1}$, therefore $V_\chi^{q+1}(G)$ is meaningful for any $\chi \in I(G)$.

In the following we write the main results of this section. We discuss the question of when the symmetry classes of tensors associated with $G = PSL_2(q)$ are nonzero vector spaces. If $\dim V = s = 1$, then it is clear that for all χ, $\chi \in I(G) - \{1_G\}$, $V_\chi^{q+1}(G) = 0$ and $V_{1_G}^{q+1}(G) \neq 0$. Therefore we deal with the case $\dim V = s = 2$ in the following theorem.

Theorem 4.1 *Consider $G = PSL_2(q)$ as a subgroup of $\$_{q+1}$ and let V be a vector space over the complex field $\mathbb{C}$, such that* $\dim V = s = 2$.
a) *If q is odd, $q = p^n$, $q \overset{4}{\equiv} 1$; then for all χ, $\chi \in I(G) - \{\chi_i, \xi_1, \xi_2 |\ i = 2, 4, \ldots, (q-5)/2;\ i \overset{4}{\equiv} 2\}$, we have $V_\chi^{q+1}(G) \neq 0$. Additionally if $q \overset{8}{\equiv} 1$, then $V_{\xi_1}^{q+1}(G) \neq 0$ and $V_{\xi_2}^{q+1}(G) \neq 0$,*
b) *If q is odd, $q = p^n$, $q \overset{4}{\equiv} 3$; then for all χ, $\chi \in I(G) - \{\theta_j, \eta_1, \eta_2 |\ j = 2, 4, \ldots, (q-3)/2;\ j \overset{4}{\equiv} 0\}$, we have $V_\chi^{q+1}(G) \neq 0$. Additionally if $q \overset{8}{\equiv} 3$, then $V_{\eta_1}^{q+1}(G) \neq 0$ and $V_{\eta_2}^{q+1}(G) \neq 0$,*
c) *If q is even, $q = 2^n$; then for all χ, $\chi \in I(G) - \{\theta_j |\ 1 \leq j \leq q/2\}$, we have $V_\chi^{q+1}(G) \neq 0$.*

Theorem 4.2 *Consider $G = PSL_2(q)$ as a subgroup of $\$_{q+1}$ and let V be a vector space over the complex field $\mathbb{C}$, such that* $\dim V = s \geq 3$. *Then for all χ, $\chi \in I(G)$, we have $V_\chi^{q+1}(G) \neq 0$.*

5 On the Orthogonal Basis and Non-Vanishing of the Symmetry Classes of tensors Associated with a Group G with Cayley Representation

Abstract

By Cayley's theorem, any finite group G of order n can be regarded as a subgroup of the symmetric group $\$_n$. Let χ be any irreducible complex character of G and let $V_\chi^n(G)$ denote the symmetry classes of tensors associated with G and χ. In this section assuming the Cayley representation of G, we obtain a formula for the dimension of $V_\chi^n(G)$ and discuss its non-vanishing in general. A necessary condition for the existence of the O-basis for $V_\chi^n(G)$ is also obtained. In the following we show some results appeared in [5].

Let V be an m-dimensional vector space over the complex field $\mathbb{C}$ and let G be a finite group and Ω be a set of n elements. Suppose G acts faithfully on Ω, so we can assume that G is a subgroup of $\$_n$, i.e., $G = \{g|\ g \in G\} = \{\sigma_g|\ g \in G\}$ where $\sigma_g :\ \Omega \to \Omega$ is defined by $\sigma_g(\omega) = g.\omega$ for all $\omega \in \Omega$, is a permutation on n letters. Therefore the vector space $V_\chi^n(G)$ is meaningful for all $\chi \in I(G)$. The following results are main results of this section.

Theorem 5.1 *Let G be a group of order n, that is, a subgroup of $\$_n$ by Cayley representation. If V is an m-dimensional vector space over the complex field $\mathbb{C}$, then for all $\chi \in I(G)$, we have*

$$\dim V_\chi^n(G) = \frac{\chi(1)}{n} \sum_{g \in G} \chi(g) m^{n/o(g)},$$

in particular, for all $m \geq 2$, $V_\chi^n(G) \neq 0$.

Theorem 5.2 *Let G be a non-trivial group of order n, that is, a subgroup of $\$_n$ by Cayley representation. If V is an m-unitary space, $m \geq 2$, and $\chi \in I(G)$ such that $\chi(1)^2 > |G|/2$, then $V_\chi^n(G)$ dose not have an O-basis.*

References

[1] L. J. Cummings, *Cyclic Symmetry Classes*, J. Algebra **40** (1976), 401-405.

[2] M. R. Darafsheh, M. R. Pournaki, *On the Dimensions of Cyclic Symmetry Classes of Tensors*, J. Algebra **205** (1998), no. 1, 317-325.

[3] M. R. Darafsheh, M. R. Pournaki, *On the Orthogonal Basis of the Symmetry Classes of Tensors Associated with the Dicyclic Group*, Linear and Multilinear Algebra **47** (2000), no. 2, 137-149.

[4] M. R. Darafsheh, M. R. Pournaki, *Computation of the Dimensions of Symmetry Classes of Tensors Associated with the Finite two Dimensional Projective Special Linear Group*, Appl. Algebra Engrg. Comm. Comput. **10** (2000), no. 3, 237-250.

[5] M. R. Darafsheh, M. R. Pournaki, *Non-Vanishing and Orthogonal Basis of Symmetry Classes of Tensors*, To Appear in Southeast Asian Bull. Math.

[6] R. Freese, *Inequalities for Generalized Matrix Functions Based on Arbitrary Characters*, Linear Algebra Appl. **7** (1973), 337-345.

[7] R. R. Holmes, T. Y. Tam, *Symmetry Classes of Tensors Associated with Certain Groups*, Linear and Multilinear Algebra **32** (1992), 21-31.

[8] I. M. Isaacs, *"Character Theory of Finite Groups"*, Academic Press, New York, 1976.

[9] M. Marcus, *"Finite Dimensional Multilinear Algebra"*, Part 1, Marcel Dekker, New York, 1973.

[10] R. Merris, *The Dimension of Certain Symmetry Classes of Tensors II,* Linear and Multilinear Algebra **4** (1976), 205-207.

[11] R. Merris, *"Multilinear Algebra"*, Gordon and Breach Science Publishers, 1997.

[12] R. Merris, M. A. Rashid, *The Dimension of Certain Symmetry Classes of Tensors*, Linear and Multilinear Algebra **2** (1974), 245-248.

[13] T. Y. Tam, *On the Cyclic Symmetry Classes*, J. Algebra **182** (1996), 557-560.

[14] B. Y. Wang, M. P. Gong, *The Subspace and Orthonormal Bases of Symmetry Classes of Tensors*, Linear and Multilinear Algebra **30** (1991), 195-204.